There Is No Really Good Definition of Mass

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There seems to be a fairly prevalent belief in the physics community that the basic concepts of our discipline (mass, force, energy, and so forth) are well understood and easily defined.[1] After all, there are dozens of textbooks on every level that supposedly define all the terms they introduce. Apparently, we teachers can pass this wisdom on to our students without any cautionary notes and without any concern for subtleties. Remarkably, this is most certainly not the case, and anyone who has studied the foundational literature in physics over the last several centuries knows that none of the fundamental ideas is satisfactorily defined.

To illustrate that point, we will show that today's leading textbook definition of *mass* (which is directly derived from the work of Ernst Mach) is fatally flawed. Furthermore, we will establish for totally different reasons arising from special relativity that Mach's own definition is erroneous, and one cannot even return to it for salvation. At the present time the idea of a rigorous operational definition of mass is an illusion, and everyone teaching physics on any level should know as much.

Definitions in General

Broadly speaking, there are two types of definition used in physics: The traditional *conceptual definition*, which specifies the meaning of an idea in terms of other, more fundamental notions, and the somewhat newer *operational definition*, which specifies a procedure by which the concept can be measured. In the latter approach, what "mass" *is* need not be grappled with beyond maintaining that "it is that which is measured in the following way."

An inherent problem with conceptual definitions is that the most fundamental ideas are not expressible in terms of still more basic, previously defined concepts. Hence, though the notion of speed (more precisely, average speed) can be specified in terms of distance traveled and time elapsed, "What *is* distance?" Any attempt to arrive at the meaning of "distance" will unavoidably bring us to the daunting question, "What *is* space?" And once you answer that, "What *is* time?" These intractable epistemological issues have challenged philosophers for millennia.

Over the past century or so, textbooks have tended to define mass conceptually in either of two ways: One is simplistically as the "quantity of matter,"[2] and the other is in terms of inertia. In the latter case, *mass is a measure of an object's ability to resist changes in its motion*. Notice that neither of these formulations tells us how to measure mass, and so both are profoundly deficient. No matter how true the inertial definition might be, it's more metaphysical than physical because we cannot use it to directly determine the numerical value of the mass of any object. Consequently physicists today, driven by the imperative to quantify, most often rely on operational definitions. These hearken back to the seminal work of Mach (1838–1916). It is the inherent weakness of that methodology (which has not been pointed out before), and of today's textbook definitions based upon it,[3] that we are concerned with here.

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A Brief History of Mass

In the 13th century the theologian Aegidius Romanus, while considering the Eucharist, suggested that in addition to weight and volume there was a third measure of matter, the *quantitas materiae* or "quantity of matter." By the 17th century the word *mass* (*massa* in Latin) was being used to mean *quantitas materiae*. Kepler was the first to articulate the notion of inertial mass.[4] He also related the masses of objects to the mutual gravitation they experience. Soon after that, Newton formalized the dynamical context of mass. But the idea was still quite nebulous; there was not yet even a unit for mass, and Newton had to work with ratios. His definition of mass in terms of volume and density (for which there also was no unit) left much to be desired.[5]

In the 1800s Mach, along with several other people (notably Saint-Venant, Hertz, Poincaré, and Kirchhoff), realized the unsatisfactory nature of the metaphysical underpinnings of physics. Mach and others objected to defining mass as quantity of matter, a concept he saw to be "quite useless."[6] Instead (1867) he proposed a new definition,[7] indeed a whole new method of definition, based on measurement. Mach's procedure was later philosophically elaborated by Nobel Laureate Percy W. Bridgman[8] in the 1930s; thereafter it came to be known as *operationalism*. Today, although operationalism has lost some of its luster, one can find what seems to be hundreds of textbooks[3, 9] and journal articles[10] the world over that define mass much in the manner of Mach.

Mach's Definition

We needn't elaborate all the details of Mach's original development. Those can be found elsewhere[7] and do not affect the shortcomings that render the approach all but impotent. Accordingly, consider two bodies A and B having masses m_A and m_B. These bodies interact (magnetically, electrically, gravitationally; it doesn't matter how) and either attract or repel. Mach was committed to downplaying the role of force and so he replaced Newton's second law with his own "First Experimental Proposition: Bodies set opposite each other induce in each other ... contrary accelerations in the direction of their line of junction."[7] Once allowed to move, the ratio of the masses of the two bodies $m_{A/B} = m_A/m_B$ is equal to the negative ratio of their ensuing accelerations, a_A and a_B:

$$m_{A/B} = -\,a_B/a_A\,. \qquad (1)$$

The minus sign comes in because the accelerations are oppositely directed. Set the mass of either body, say B, equal to the standard unit mass (m_B = 1 kg). Then

$$m_A = -\,a_B/a_A\,. \qquad (2)$$

Simply push two repelling bodies together, let them subsequently fly apart, simultaneously measure both accelerations, and there you have it, a straightforward operational definition of mass.

Mach slipped past the "insurmountable" difficulty (to use Poincaré's words) of first defining force by avoiding Newton's first law, replacing the second, and assuming the validity of the third. Nor is it clear how Mach would have actually determined the accelerations. **Instantaneous acceleration is a mathematical idealization that is not amenable to direct measurement.**[11] Furthermore, Mach contended that objects A and B could interact without being affected by any other objects in the universe. How that might be arranged in a real Earth-bound laboratory is not obvious.

Nowadays people tend to overlook the shortcomings considered above and simply accept the basic veracity of Mach's approach. To the contrary, we will point out a fundamental flaw that cannot be overlooked, namely, that mass is interaction dependent. Mach wasn't troubled by the insights of relativity;[12] he had published his definition long before the special theory (1905) and never did accept it anyway.

Relativity & Mass

What makes Mach's definition erroneous is that the mass of an object can change when it does work, or when work is done on it as a whole or on any part thereof. To see that, imagine a free particle of mass m. Its relativistic total energy (E) is the sum of its rest energy (E_0) and its kinetic energy (KE): $E = E_0 + KE$. In the most up-to-date formulation of relativity, the one Einstein came to embrace in his later work, mass is taken to be Lorentz invariant (i.e., the same for all inertial observers).[13] Mass is, accordingly, not an explicit function of speed. The once-commonplace speed-dependent *relativistic mass* is dispensed with—the only mass is what used to be called the "rest" or "proper mass." Consequently, rest energy is given by $E_0 = mc^2$. An increase in the particle's speed produces a gain in KE (not mass), and E increases: $E = mc^2 + KE$.

Consider a composite system of mass M consisting of two or more interacting particles.[14] The system's total energy, measured in the center-of-mass frame (where it is motionless), is $E = E_0 = Mc^2$. This is the internal or rest energy of the composite entity and it's the sum of the individual rest energies ($m_i c^2$), kinetic energies (KE_i), and potential energies (PE_i) of all of the particles :

$$E = Mc^2 = \sum_i m_i c^2 + \sum_i KE_i + \sum_i PE_i \, . \tag{3}$$

Thus (with the addition of thermal energy) a kilogram of ice melts into more than a kilogram of water, increasing by about four parts in 10^{12}. Similarly, a stretched spring has more mass than it had before work was done on it. These conclusions, though they fly in the face of the traditional notion of the constancy of mass, have long been widely accepted, even if only rarely mentioned in the classroom.[15]

Incidentally, because both interpretations of relativity (i.e., wherein m is, or is not, Lorentz invariant) allow the mass of an otherwise unaltered body to change, **equating mass to *quantity of matter* is totally untenable.**[2]

When attracting particles come together to form a bound system—for example, an atom or a nucleus, or even a planet[16]—the mass of the whole (M) will be less than the sum of the masses of the constituents. A deuteron, having a mass defect of 0.00239 u, has that much less mass (i.e., 3.97×10^{-27} g or about four electron masses) than the free neutron and proton that melded under the strong interaction to form it.[17] It's reasonable to conclude that the neutron and proton that persist, bound together as the deuteron, must each have lost mass.

As long as there is an attractive interaction between them, objects are generally more massive apart than together. The opposite is true for a system of repulsive particles; as they come together they gain potential energy and, equivalently, gain mass. Mass, although Lorentz invariant, is interaction dependent.[18]

Let's reenact Mach's definition using two magnetized objects of mass m_A and m_B. Suppose these are far apart, at rest on an air table in a laboratory. If they repel each other, and we bring them together, their initial composite mass becomes $M = (m_{Ai} + m_{Bi})$, and this is greater than the sum of their individual "free" essentially zero-PE masses ($m_A + m_B$). The masses of the objects when both are at rest are proportional to their rest energies; because they are interacting they possess an additional amount of mass equal to PE/c^2. The very act of setting up the experiment by doing positive work on the objects increases their overall mass.

Allowed to move, the two objects accelerate away from each other. For both A and B, mass (via potential energy) decreases as kinetic energy increases—their total energy is constant. Following Mach, we assume (presumptuously) that we can measure the instantaneous accelerations[11] of A and B simultaneously at some arbitrary final moment. Setting the mass of B, m_{Bf}, equal to 1 kg, we then supposedly determine the mass of A, m_{Af}, at that time.

At the end of the run, when the objects are brought to rest where they started (by taking out the energy we put in initially), their masses will again be m_A and m_B, where $m_{Af} > m_A$ and $m_{Bf} > m_B \neq 1$ kg. And this would be the case using either interpretation of mass (Lorentz invariant or relativistic), although the details differ. Thus we have failed to measure either m_A or m_B; **Mach's procedure does not provide a practical operational definition of mass.**[19]

Today's Textbook Definitions

The leading introductory physics texts[3] all handle the definition of mass in the same unsatisfactory way. They first define force: the version given by Halliday, Resnick, and Walker is representative, "An interaction that causes an acceleration of a body is called a **force**...." A student might well ask, "What then is an interaction?" Is it not composed of two equal, oppositely directed forces? As a slight variation, Tipler[3] offers us, "A **force** is an influence on an object that causes the object to change its velocity, that is, to accelerate." At that point our student might properly ask, "What is an influence?" And suppose this "influence" only contributes to the deformation of the object without accelerating it; is it still a force? Is your weight a force while you are standing motionlessly on a scale?

Giancoli[3] tells us that “Intuitively, we experience **force** as any kind of a push or a pull on an object.” Here one must assume that a “push” is a compressive force and a “pull” is a tensile force. Any attempt to use these two, *always undefined*, terms to define force (as is still too often done[20]) is surely misleading and tautological at best. We cannot replace the one undefined word “force” with the two undefined words “push” and “pull,” no matter how experientially familiar they are.

However wanting these textbook attempts at defining force are, they are all conceptual definitions, not operational ones. Overlooking their failings, it would nonetheless be completely inappropriate to base an operational definition of mass on a non-operational definition of force. But that is precisely what all of these books do.

It is next suggested by every one of these texts[3] that a force be applied to body-1 of mass m_1 and the resulting acceleration, a_1, be measured. Exactly how this force is to be supplied is left to the imagination, as if it were of no concern. “The same force” is then applied to body-2 of mass m_2 and its acceleration, a_2, is measured. Assuming that $F = ma$, the ratio of these accelerations is the ratio of the masses and we are finished; we have a definition of mass, à la Mach. But this is not what Mach suggested, and it brushes past an irrresolvable difficulty (avoided by Mach), one pointed out by Poincaré.[21] Halliday et al. seem to be the only ones who suspect that they are on terribly soft ground here, because they at least write, “We next apply that same force (we would need some way of being certain it is the same force) to a second body....” Not having an operational definition of force and therefore not knowing how to measure it, and moreover, not yet having defined mass, there is simply no way “of being certain it is the same force.” As Poincaré put it, “A force applied to a body cannot be uncoupled and applied to another body as an engine is uncoupled from one train and coupled to another. It is therefore impossible to say what acceleration such a force, applied to such a body, would give to another body if it were applied to it.”

Unlike Mach, James Clerk Maxwell (1877) did not shun force; in fact he gave it priority over mass. Still, rather than explicitly defining force, Maxwell conveniently asserted that force was “completely defined and described in Newton’s three laws of motion.”[22] [That’s not really true since Newton was very careful to first provide separate definitions of several kinds of force (viz., his Definitions 3, 4, 5, 6, 7, and 8).] Maxwell then suggested that the extension of an elastic thread (or of a spring) tied to a body could be used to specify the force exerted via that thread.

According to Maxwell, the same extension of the thread would produce the same applied force on different bodies. This assumes that everything will remain exactly the same during all trials, and that’s unrealistic. One can anticipate that stretching an elastic thread would result in residual internal changes that might affect its subsequent behavior—no material is perfectly elastic. Any macroscopic elastic system will become altered, however slightly, as it’s cycled; real materials contain flaws and experience internal friction, temperature variations, atomic slippage, deformation, fatigue, and so forth. There is no way to know for a fact that a particular elastic thread will exert exactly the same force during successive applications. Nor is it clear how one might unambiguously test Maxwell’s assumption. That’s why Mach worked with two objects simultaneously (making use of the “action” and equal “reaction”) rather than this sequential procedure, which looks simpler but is far more problematic.

Any attempt to bypass these impediments by defining force statically via weight raises a host of robust difficulties.[10, 21] Of these, not the least is the necessity to operationally define gravitational mass[23] and to establish its equivalency with inertial mass. That aside, weight is a poor notion to deal with experimentally. On Earth it’s only crudely predictable and is not constant in time. There isn’t even universal agreement on the meaning of the word.[24]

Nonetheless, suppose we take Maxwell’s idealized elastic thread, attach it to our textbook mass, apply a horizontal force (F) at the far end, keep the extension precisely constant, and measure the acceleration.[25] Poincaré maintained that we cannot know if F applied to the thread is transmitted to the body without first assuming Newton’s third law, which we haven’t yet established experimentally. (The same is true for a static weight measurement using a thread or spring.) Moreover, any real “horizontal” thread, spring, rod, or wire will sag under its own weight and the force applied to the body will not be horizontal. The weight of the sagged thread itself will provide a horizontal force component, which, since everything is frictionless, will accelerate the object even without an externally applied force.

To deal with these issues let's move our lab to the mythical "weightless" reaches of outer space. Still, a thread has mass (m_t) and that will add to the mass of the body (m_1). If F is applied to the end of the thread in a frictionless experiment, acceleration (a_1) will result. The thread and body-1 will accelerate together such that $F = (m_t + m_1)a_1$. When the same force is subsequently applied to body-2 (assuming that can be done), $F = (m_t + m_2)a_2$. To be completely rigorous m_t cannot be neglected, and the ratio of the resulting accelerations is not equal to the ratio of the masses of the two bodies. If we could measure instantaneous acceleration, remove all outside influences, work in a weightless environment, and apply a perfectly constant reproducible force at will (in the manner of Maxwell)—and we cannot manage *any* of these!—the standard textbook scheme would still be useless as a definitional procedure.

Conclusion

The mass of an object depends on where it is in relation to the other entities with which it interacts. This effect cannot be ignored if we are to create a completely correct operational definition of mass. As a consequence, Mach's definition fails, and the myriad books and articles that have unquestioningly embraced it over the past hundred years or so are perforce in error. For entirely different reasons, the consensus definition provided by most contemporary introductory textbooks, which is an ill-conceived "simplification" of Mach's procedure, is essentially useless and in its glibness, inappropriately misleading.

Notes & References

1. Art Hobson, "Energy and work," letter to the editor, *Phys. Teach.* **42**, 260 (May 2004).
2. Digby Swift, *Physics for GCSE* (Basil Blackwell Ltd., Oxford, 1988), p. 8. Neville Warren, *Excel HSC Physics* (Pascal Press, Glebe NSW, 2001), p. 328.
3. Raymond A. Serway and Robert J. Beichner, *Physics for Scientists and Engineers* (Saunders College Pub., Philadelphia, PA, 2000), p. 116. Paul A. Tipler, *Physics for Scientists and Engineers* (Worth Publishers, New York, 1991), p. 81. Hugh D. Young and Roger A. Freedman, *University Physics* (Addison-Wesley, Reading, MA, 1996), p. 101. Douglas C. Giancoli, *Physics for Scientists and Engineers* (Prentice Hall, Upper Saddle River, NJ, 2000), p. 82. David Halliday, Robert Resnick, and Jearl Walker, *Fundamentals of Physics* (Wiley, New York, 1997), p. 83.
4. According to Kepler, "Inertia or opposition to motion is a characteristic of matter; it is stronger, the greater the quantity of matter in a given volume." Max Jammer, *Concepts of Mass in Classical and Modern Physics* (Dover Publications Inc., Mineola, NY, 1997), p. 56.
5. Newton's Definition 1 is "Quantity of matter is a measure of matter that arises from its density and volume jointly." And "Furthermore, I mean this quantity whenever I use the term 'body' or 'mass' in the following pages." Isaac Newton, *The Principia* translated by I. Bernard Cohen and Anne Whitman (University of California Press, Berkeley, 1999), p. 403. Since volume and density (i.e., relative density) are readily measured, it is likely that Newton was actually giving us an operational definition, albeit one that equated gravitational mass (via the usual floating-object methods of determining density) with inertial mass.
6. Ernst Mach, *The Science of Mechanics* (Open Court Pub. Co., Chicago, IL, 1960).
7. Max Jammer, *Concepts of Mass in Classical and Modern Physics* (Dover Publications Inc., Mineola, NY, 1997), pp. 91–103, and R. Dugas, *A History of Mechanics* (Editions du Griffon, Switzerland, 1955), pp. 443–444. E. Mach, "Uber die definition der masse," *Carl's Repertorium der Experimentalphysik* **4**, 355–359 (1868).
8. P.W. Bridgman, *The Nature of Physical Theory* (Wiley, New York, 1964), first published in 1936. See the Introduction in A.S. Eddington, *The Mathematical Theory of Relativity* (Cambridge at the University Press, London, G.B., 1924), p. 1-7, first published in 1923. The origins of operationalism have roots in the early measurements of distance and time.
9. R.B. Lindsay, *Physical Mechanics* (D. Van Nostrand Co. Inc., New York, 1933), pp. 15–18.
10. P.A. Goodinson and B.L. Luffman, "On the definition of mass in classical physics," *Am. J. Phys*. **53**, 40–42 (Jan. 1985). L. Eisenbud, "On the classical laws of motion," *Am. J. Phys.* **26**, 144–159 (March 1958). R. Weinstock, "Laws of classical motion: What's *F*? What's *m*? What's *a*?" *Am. J. Phys*. **29**, 698–702 (Oct. 1961).
11. Any experimental determination of instantaneous acceleration must use a finite change in velocity ($\Delta\mathbf{v}$) occurring during a finite time interval (Δt). Real measurements, no matter how sophisticated, can only yield an average acceleration computed over a small but finite time interval. (The same is true for **v**.) Ironically, only when the acceleration is constant can instantaneous acceleration be determined experimentally, but no measured acceleration can be proven to be exactly constant (over any time scale, however small).
12. In relativity acceleration does not necessarily occur along the line of action of the force. If **F** is perpendicu-

lar to **v**, then **F** = γm**a**, whereas if **F** is parallel to **v**, then **F** = $\gamma^3 m$**a**. This suggests that "inertial mass" is subtler than generally assumed. See L.B. Okun, "The concept of mass (mass, energy, relativity)," *Sov. Phys. Usp*. **32**, 629 (1989).

13. W.G.V. Rosser, *Introductory Special Relativity* (Taylor & Francis, London, 1991), p. 18. Carl G. Adler, "Does mass really depend on velocity, dad?" *Am. J. Phys.* **55**, 739–743 (Aug. 1987). Gordon J. Aubrecht II, "Comment on 'Apparatus to measure relativistic mass increase,'" *Am. J. Phys.* **72**, 970–971 (July 2004). T.R. Sandin, "In defense of relativistic mass," *Am. J. Phys*. **59**, 1032–1036 (Nov. 1991). For a comparison of the two conventions, see R.P Bickerstaff and G. Patsakos, "Relativistic generalizations of mass," *Eur. J. Phys.* **16**, 63 (1995). For a bibliography, see L.B. Okun, "Note on the meaning and terminology of special relativity," *Eur. J. Phys.* **15**, 403 (1998).
14. Anna Beck, *Collected Papers of Albert Einstein*, *Vol. 2* (Princeton University Press, Princeton, NJ, 1989), doc. 47, p. 286. Leo Sartori, *Understanding Relativity* (University of California Press, Berkeley, CA, 1996), p. 206.
15. Albert Einstein and Leopold Infeld, *The Evolution of Physics* (Simon and Schuster, New York, 1938), p. 208. Robert Resnick and David Halliday, *Basic Concepts in Relativity and Early Quantum Theory* (Macmillan, New York, 1992), p. 111. A.P. French, *Special Relativity* (W.W. Norton & Co., New York, 1968), p. 20. Francis W. Sears and Robert W. Brehme, *Introduction to the Theory of Relativity* (Addison-Wesley, Reading, MA, 1968), p. 127. E.P. Ney, *Electromagnetism and Relativity* (Harper & Row, New York, 1962), p. 68.
16. Eugene Hecht, "An historico-critical account of potential energy: Is *PE* really real?" *Phys. Teach*. **41**, 486–493 (Nov. 2003).
17. T.S. Mudhole and N. Umakantha, "Binding energy of the deuteron: A laboratory experiment," *Am. J. Phys.* **43**, 104–105 (Jan. 1975).
18. "Invariance means 'same value as determined from different inertial frames of reference,' *not* 'unchanged by impact or external forces.'" Edwin F. Taylor and John Archibald Wheeler, *Spacetime Physics* (W.H. Freeman, San Francisco, CA, 1966), p. 134.
19. Saint-Venant (1851) considered two objects that spring apart after they collide. Provided they do not subsequently interact, the ratio of their constant parting speeds equals the ratio of their masses. It's still often suggested [e.g., Jay Orear, *Physics* (Macmillan, New York, 1979), p. 56 or W.L.H. Shuter, "Mechanics experiments using modified PSSC apparatus" *Am. J. Phys.* **13**, 766 (Oct. 1965)] that this scheme be used to define mass operationally. Unfortunately, real objects do interact gravitationally, and even in a frictionless environment their instantaneous recoil speeds would vary in time, as would their masses.
20. Paul Hewitt, *Conceptual Physics* (Addison-Wesley, Reading, MA, 1998), p. 61. For a list of texts that uncritically define force as pushes and pulls see J.S. Touger, "When words fail us," *Phys. Teach.* **29,** 90–95 (Feb. 1991).
21. Henri Poincaré, *The Value of Science* (The Modern Library, New York, 2001), p. 78, first published in 1903.
22. James Clerk Maxwell, *Matter and Motion* (Dover Pub. Inc., New York, 1991), pp. 26–35. Contrast this with James L. Anderson, "Newton's first two laws of motion are not definitions," *Am. J. Phys.* **58,** 1192–1195 (Dec. 1990).
23. In general relativity there is no gravitational force per se, and this suggests that the notion of "gravitational mass" is subtler than we ordinarily assume.
24. If "weight" is the gravitational attraction between an object and a nearby celestial body (usually, the Earth), then we have to distinguish that from "effective weight," which is measured by a scale. Since our planet's spin is slowing, things everywhere but at the poles are getting heavier with time. The Sun, Moon, and other planets exert a small gravitational force on terrestrial objects, making an operational definition of weight much more complicated. Chester H. Page, "What is weight?" *Am. J. Phys.* **43,** 920–921 (Oct. 1975) and Frank O. Goodman, "Weightful versus weightless," *Am. J. Phys*. **62**, 775 (Sept. 1994). Alternatively, A.P. French "On weightlessness," *Am. J. Phys*. **63**, 105–106 (Feb. 1995).
25. Satinder S. Sidhu, "Pristinely pure law in the lab," *Phys. Teach*. **32**, 282–283 (May 1994). F.E. Domann, "An improved Newton's second law experiment," *Am. J. Phys.* **50**, 185–186 (Feb. 1982). These experiments are typical of most that are carried out in the introductory physics lab. They make assumptions that are not strictly true (e.g., force is constant) and then "prove" what they set out to prove.

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