Gaussian solitons in nonlinear Schrödinger equation(*)

A. B. Nassar(1)(2), J. M. F. Bassalo(2), P. T. S. Alencar(2), J. F. de Souza(2)
J. E. de Oliveira(3) and M. Cattani(4)

(1) *Extension Program-Department of Sciences, University of California - Los Angeles California 90024*
Physics Department, The Harvard-Westlake School - 3700 Coldwater Canyon N. Hollywood, CA 91604
(2) *Departamento de Física da UFPA - 66075-900, Guamá, Belém, Pará*
(3) *Centro Federal de Educação Tecnológica do Pará - 66060-600, Marco, Belém, Pará*
(4) *Instituto de Física da USP - C.P. 66318, 05315-970, São Paulo, SP*

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Summary. — We find a condition on the parameter controlling the strength of the nonlinearity of a nonlinear Schrödinger equation which grants the possibility of nonspreading Gaussian wave packet solutions for an inverted parabolic potential. Our analysis is performed using the de Broglie-Bohm formalism.

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1. – Introduction

As pointed out in a recent paper [1], in the 21th century, information technology will become very important to realize a worldwide communication network. Optical transmission using short optical pulse train is a fundamental technology for achieving a high-speed and long-distance global network. Among many optical transmission formats, an optical soliton, which is created by balancing the anomalous group velocity dispersion with the fiber nonlinearity, called the self-phase modulation, offers a great potential to realize an advanced optical transmission system since the soliton pulse can maintain its wave form over long distances. So, it is extremely important, from the experimental and theoretical points of view [1], to investigate the existence and stability of solitons under special conditions, that is, parameter functions describing dispersion, nonlinearity, absorption and other properties that there are present in the signal transmission. The existence and analytical stability of solitons have been extensively studied, for instance, within the framework of nonlinear Schrödinger and Dirac-type equations [1-9]. Studies

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done by Nakazawa *et al.* [1] suggests that it could exist solitonlike steady-state pulse when the governing equation is a NLSE with a parabolic potential. However, since their analysis has been performed taking into account also dispersion effects, it is difficult to clearly understand the essential contribution of the parabolic potential to the solitons stability.

As is well known [1], there are soliton solutions to the NLSE for a constant external field. The existence of soliton solutions in other potential energies, different from a constant field has remained, for along time, an open question. According to Hasse [10], it would be interesting to investigate the existence of solitons for a NLSE with an inverted parabolic potential. We found a condition on the parameter controlling the strength of the nonlinearity of the NLSE which grants the possibility of nonspreading Gaussian wave packet solutions of the NLSE.

2. – Nonlinear Schrödinger equation and the De Broglie-Bohm formalism

Many nonlinear equations have soliton solutions characterized by widths which do not spread in time. The solitons of great interest to energy transport have been the nondispersing wave packet solutions of the nonlinear Schrödinger equation (NLSE):

$$(1)\qquad i\,\hbar\frac{\partial\psi(x,\,t)}{\partial t} = -\,\frac{\hbar^2}{2\,m}\,\frac{\partial^2\psi(x,\,t)}{\partial x^2} + V_\epsilon\,\psi(x,\,t) - G\mid\psi(x,\,t)\mid^2\psi(x,\,t)\,,$$

where m is the effective mass of the excitation, V_ϵ is a constant average potencial energy, and G is the parameter controlling the strength of the nonlinearity [10-13]. This ubiquitous equation possesses a well-known wave packet soliton solution in terms of a hyperbolic secant function:

$$(2)\qquad \psi(x,\,t) = \left(\frac{k}{2}\right)^{1/2}\operatorname{sech}\left[k(x - x_0 - v\,t)\right]\,\exp\left[\frac{i\,m\,v}{\hbar}\,(x - x_0\,v'\,t)\right]\,,$$

where $k = (m\,G/2\,\hbar^2)$ and $v' = [V_\epsilon + (m/2)\,v^2 - \hbar^2\,k^2/2]/m\,v$. This solution represents a nonspreading wave packet initially centered at x_0 and moving along a classical trajectory with constant velocity v.

There exist also soliton solutions to eq. (1) for a constant external field [10]. These solitons move with constant acceleration and have the same shape as their counterpart without the external field. Whether there exist soliton solutions in other potential energies other than for a constant field has remained an open question. As suggested by Hasse [10], it would be interesting to possibly find solitons for the NLSE sliding down an inverted parabolic potential.

In this work, we answer the aboved-posed question by finding the conditions on the parameter G which grant the possibility of nonspreading Gaussian wave packet solutions to eq. (1) for an inverted parabolic potential. To this end, we develop a semiclassical method for wave packets within the de Broglie-Bohm [14-18] formulation of quantum mechanics and the theory of invariants. The notion of trajectories within this formalism is kept and the differences between the quantum and classical trajectories are exhibited. This method can incorporate nonlinear semiclassical dynamical information and display a direct connection between the macroscopic and microscopic levels of the problem.

We begin by expressing the wave function in the form $\psi = \phi \exp[i\, S/\hbar]$, which transform eq. (1) into [14-18]

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0\,, \tag{3}$$

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} = -\frac{1}{m}\frac{\partial}{\partial x}\left(V_\epsilon + V_{n\ell} + V_{qu}\right). \tag{4}$$

Equation (3) represents the conservation of probability with density $\rho = \phi^2$, and whereas eq. (4) describes trajectories of a particle with velocity $v = (1/m)(\partial S/\partial x)$ subject to an arbitrary external potential V_ϵ, the nonlinear potential

$$V_{n\ell} = -G\,\phi^2 \tag{5}$$

and the so-called quantum potential

$$V_{qu} = -\frac{\hbar^2}{2\,m\,\phi}\frac{\partial^2 \phi}{\partial x^2}\,, \tag{6}$$

which provides the connection between the quantum and classical levels. The quantum trajectories are defined by the de Broglie guidance equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v(x,\,t)\,\big|_{x=x(t)} = \frac{1}{m}\frac{\partial S}{\partial x}\,\big|_{x=x(t)}\,. \tag{7}$$

With the help of eqs. (4) and (7), we have

$$\hbar\,\frac{\partial S}{\partial t} + \frac{1}{2\,m}\left(\frac{\partial S}{\partial x}\right)^2 + \left(V_\epsilon + V_{n\ell} + V_{qu}\right) = 0\,. \tag{8}$$

Expansion of the quantum potential around the classical trajectory $X(t)$ yields

$$V_{qu}(x,t) = V_{qu}\,[X(t),t] + V'_{qu}[X(t),\,t]\,[x - X(t)] + \\ + \frac{V''_{qu}[X(t),t]}{2}\,[x - X(t)]^2 + \ldots. \tag{9}$$

The semiclassical approximation can be obtained by setting the second term in the expansion (the quantum force) to zero, *i.e.* $F_{qu}\,[x - X(t)] = 0$, which allows the quantum potential to be written explicitly:

$$V_{qu}(x) = -\frac{\hbar^2}{2\,m}\left(\frac{[x - X(t)]^2}{4\,a^4(t)} - \frac{1}{2\,a^2(t)}\right), \tag{10}$$

which after integration yields

$$\rho(x,t) = [2\,\pi\,a^2(t)]^{-1/2}\exp\left[-\frac{[x - X(t)]^2}{2\,a^2(t)}\right]. \tag{11}$$

Substitution of eq. (11) into (4) gives

(12) $$v(x,t) = \frac{\dot{a}(t)}{a(t)}\,[x - X(t)] + \dot{X}(t)\,.$$

The semiclassical description of the wave packet dynamics can be accomplished by expanding $S(x,\ t)$, $V_\epsilon(x,\ t)$, and $V_{n\ell}(x,\ t)$ around $X(t)$, namely

(13) $$S(x,t) = S\,[X(t),t] + S'[X(t),\ t]\,[x - X(t)] + \frac{S''[X(t),t]}{2}\,[x - X(t)]^2 + \ldots\,,$$

(14) $$V_\epsilon(x,t) = V_\epsilon\,[X(t),t] + V_\epsilon'[X(t),\ t]\,[x - X(t)] + \frac{V_\epsilon''[X(t),t]}{2}\,[x - X(t)]^2 + \ldots$$

(15) $$V_{n\ell}(x,t) = V_{n\ell}\,[X(t),t] + V_{n\ell}'[X(t),\ t]\,[x - X(t)] + \frac{V_{n\ell}''[X(t),t]}{2}\,[x - X(t)]^2 + \ldots\,.$$

Connection to eq. (12) can be established using eq. (8), by collecting terms in $[x - X(t)]^0$ and $[x - X(t)]$:

(16) $$S'[X(t),t] = \frac{m\,\dot{X}(t)}{\hbar}\,,$$

(17) $$S''[X(t),t] = \frac{m\,\dot{a}(t)}{\hbar\,a(t)}\,.$$

Now substituting eqs. (10), (13)-(17) into eq. (7) and collecting terms in $[x - X(t)]^2$, $[x - X(t)]^1$, and $[x - X(t)]^0$ we have the following set of nonlinear equations which describe the wave packet dynamics:

(18) $$\ddot{a}(t) + \left(\frac{1}{m}\,V_\epsilon''[X(t),t]\right)a(t) = \frac{\hbar^2}{4\,m^2\,a^3(t)} - \frac{G}{\sqrt{2\,\pi}\,m\,a^2(t)},$$

(19) $$\ddot{X}(t) + \frac{1}{m}\,V_\epsilon'[X(t),\ t] = 0,$$

(20) $$\hbar\,\dot{S}_0 = \frac{m}{2}\,\dot{X}^2(t) - V_\epsilon\,[X(t),\ t] + \frac{G}{\sqrt{2\,\pi}\,a(t)} - \frac{\hbar^2}{4\,m\,a^2(t)},$$

where we have denoted $S_0(t) = S[X(t),\ t]$ the quantum action.

The wave packet dynamics is now completely determined by eqs. (11), (18)-(20). The wave packet width and phase displays their quantum-mechanical nature through the presence of the potentials in the last two terms of eqs. (18) and (20). We impose here the following conditions on eqs. (18)-(20):

(21) $$a(0) = a(t) = a_0\,;\quad X(0) = X_0\,,\quad \dot{X}(0) = V_0\,;\quad S_0(0) = m\,V_0\,X_0\,.$$

For a Gaussian soliton sliding down an inverted parabolic potential $[V_\epsilon = -\frac{m}{2}\,\Omega^2\,X^2]$ we can show, after a straightforward, but tedious calculation, that

$$\psi(x,\,t) = \left(2\,\pi\,a_0^2\right)^{-1/4} \exp\left[-\frac{1}{4\,a_0^2}\,[x - X(t)]^2\right] \times \times \exp\left[\frac{i\,m\,\dot{X}(t)}{\hbar}\,[x - X(t)] + \frac{i\,m\,V_0\,X_0}{\hbar}\right] \times \times \exp\left[\frac{i}{\hbar}\int_0^t \mathrm{d}t'\left[\frac{m}{2}\,\dot{X}^2(t') + \frac{m}{2}\,\Omega^2\,X^2(t') + \frac{G}{\sqrt{2\,\pi}\,a_0} - \frac{\hbar^2}{4\,m\,a_0^2}\right]\right], \tag{22}$$

where $X(t) = X_0\,\cosh\,(\Omega\,t) + (V_0/\Omega)\,\sinh\,(\Omega\,t)$ and a_0 is a solution to the quartic equation

$$a_0^4 - \left(\frac{G}{\sqrt{2\,\pi}\,m\,\Omega^2}\right) a_0 + \frac{\hbar^2}{4\,m^2\,\Omega^2} = 0\,. \tag{23}$$

For this equation to have a real solution we must have

$$G > 1.56\sqrt{\frac{\hbar^3\,\Omega}{m}}\,. \tag{24}$$

This condition establishes a minimum (nonzero) value for the controlling parameter G, which can be of importance in describing the evolution of a Gaussian soliton. This answers the question posed at the beginning of this work concerning the existence of soliton solutions in other potential energies other than for a constant field. There exist Gaussian soliton solutions to the NLSE accelerating down an inverted parabolic potential and having the same shape as their counterpart without the external field provided that condition (24) holds true.

3. – Conclusions

We have shown that, when the nonlinearity strength parameter G of the NLSE, defined by eq. (1), obeys the critical condition, $G > 1.56\sqrt{\frac{\hbar^3\,\Omega}{m}}$, the wave packet solutions behave as Gaussian solitons for an inverted parabolic potential. So, our predictions confirm Hasse suggestion [10].

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