BRIEF REPORTS

Wave propagator via quantum fluid dynamics

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Via the de Broglie–Bohm causal interpretation of quantum mechanics, we develop a protocol to obtain a propagator for the guiding wave function where the features of the quantum potential are kept. Our analysis is extended to include a friction mechanism. [S1063-651X(97)09507-X]

In the causal interpretation of quantum mechanics, the primary concept is introduced that a particle has a definite path which is determined by a suitable equation of motion and that this path is fundamentally affected by a guiding wave function [1–5]. Accordingly, the connection between the particle and wave properties can be obtained by writing the guiding wave function in the polar form

$$\psi = \phi \exp(iS), \qquad (1)$$

where

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \qquad (2)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = -\frac{1}{m} \frac{\partial}{\partial x}(V + V_{qu}), \qquad (3)$$

and

$$\rho = \phi^2, \qquad (4)$$

$$v = \frac{\hbar}{m} \frac{\partial S}{\partial x}, \qquad (5)$$

$$V_{qu} = -\frac{\hbar^2}{2m\phi} \frac{\partial^2 \phi}{\partial x^2}. \qquad (6)$$

Equation (2) represents the conservation of probability with density ρ, whereas Eq. (3) describes paths of a particle with velocity

$$\frac{dx}{dt} = v(x,t)\big|_{x=x(t)} = \frac{\hbar}{m} \frac{\partial S}{\partial x}\bigg|_{x=x(t)}, \qquad (7)$$

subject to an arbitrary external potential V and the so-called quantum potential V_{qu} [1–5].

With the help of Eq. (7), we can readily obtain

$$\frac{\partial S}{\partial t} + \left(\frac{\hbar}{2m}\right)\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{\hbar}(V + V_{qu}) = 0. \qquad (8)$$

From Eqs. (7) and (8), we can also obtain

$$\frac{d^2x}{dt^2} = -\frac{1}{m} \frac{\partial}{\partial x}(V + V_{qu}), \qquad (9)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \qquad (10)$$

is the hydrodynamical derivative.

Equation (9) has the form of Newton's second law, in which the particle is subject to a quantum potential V_{qu} in addition to the classical potential V. The classical set of paths is obtained by considering the case when the amplitude of the wave function is a slowly varying function of position, i.e., $V_{qu} \rightarrow 0$.

In what follows, we develop a protocol to obtain a propagator for a wave function by retaining explicitly some of the features of the quantum potential. Therefore, this procedure attempts to generalize that developed by Feynman and Hibbs

[6], since the procedure of Feynman and Hibbs is viewed as a method for obtaining the quantum wave function from the set of classical paths, for which $V_{qu} = 0$.

We investigate the quantum hydrodynamical evolution of the wave packet

$$\phi(x,t) = [2\pi a^2(t)]^{-1/4} \exp\left\{-\left[\frac{[x-X(t)]^2}{4a^2(t)}\right]\right\}, \quad (11)$$

where $X(t)$ represents the classical path. To this end, we expand $S(x,t)$, $V(x,t)$, and $V_{qu}(x,t)$ around $X(t)$ up to second order:

$$S(x,t) = S[X(t),t] + S'[X(t),t][x-X(t)]$$
$$+ \frac{S''[X(t),t]}{2}[x-X(t)]^2, \quad (12)$$

$$V(x,t) = V[X(t),t] + V'[X(t),t][x-X(t)]$$
$$+ \frac{V''[X(t),t]}{2}[x-X(t)]^2, \quad (13)$$

$$V_{qu}(x,t) = V_{qu}[X(t),t] + V'_{qu}[X(t),t][x-X(t)]$$
$$+ \frac{V''_{qu}[X(t),t]}{2}[x-X(t)]^2. \quad (14)$$

Next, substituting Eq. (11) into Eq. (2) and integrating, we find

$$v(x,t) = \frac{\dot{a}}{a}[x-X(t)] + \dot{X}(t). \quad (15)$$

A connection to Eq. (12) can be established with the help of Eq. (7) by collecting terms in $[x-X(t)]^0$ and $[x-X(t)]$:

$$S'[X(t),t] = \frac{m\dot{X}}{\hbar}, \quad (16)$$

$$S''[X(t),t] = \frac{m\dot{a}}{\hbar a}. \quad (17)$$

Now, substituting Eqs. (11)–(17) into Eq. (8) and collecting terms in $[x-X(t)]^0$, $[x-X(t)]$, and $[x-X(t)]^2$, we have

$$\dot{S}_0 = \frac{1}{\hbar}\left(\frac{1}{2}m\dot{X}^2 - V[X(t),t] - \frac{\hbar^2}{4ma^2}\right), \quad (18)$$

$$\ddot{X} = -\frac{1}{m}V'[X(t),t], \quad (19)$$

$$\ddot{a} + \left(\frac{1}{m}V''[X(t),t]\right)a = \frac{\hbar^2}{4m^2 a^3}, \quad (20)$$

where we have denoted $S_0(t) = S[X(t),t]$. It is worth noticing the presence of the quantum potential V_{qu} in the last terms of Eqs. (18) and (20). These equations have the initial conditions

$$X(0) = x_0, \quad \dot{X}(0) = v_0,$$

$$a(0) = a_0, \quad \dot{a}(0) = 0,$$

$$S_0(0) = \frac{mv_0 x_0}{\hbar}. \quad (21)$$

Now the wave packet described by Eq. (1) can be written as

$$\psi(x,t) = [2\pi a^2(t)]^{-1/4}\exp\left[\left(\frac{im\dot{a}(t)}{2\hbar a(t)} - \frac{1}{4a^2(t)}\right)\right.$$
$$\times [x-X(t)]^2\right]\exp\left[\frac{im\dot{X}(t)}{\hbar}[x-X(t)] + \frac{imv_0 x_0}{\hbar}\right]$$
$$\times \exp\left[\frac{i}{\hbar}\int_0^t dt'\left(\frac{1}{2}m\dot{X}^2(t') - V[X(t)]\right.\right.$$
$$\left.\left.- \frac{\hbar^2}{4ma^2(t)}\right)\right]. \quad (22)$$

Next, we turn to finding the propagator $K(x,x_0,t)$ as defined by the integral equation

$$\psi(x,t) = \int_{-\infty}^{+\infty} dx_0 K(x,x_0,t)\psi(x_0,0). \quad (23)$$

Let us first define the normalized quantity

$$\Phi(v_0,x,t) = (2\pi a_0^2)^{1/4}\psi(v_0,x,t), \quad (24)$$

which satisfies the completeness relation [7]

$$\int_{-\infty}^{+\infty} dv_0 \Phi^*(v_0,x,t)\Phi(v_0,x',t) = (2\pi\hbar/m)\delta(x-x'). \quad (25)$$

From Eq. (2), it follows that

$$\frac{\partial(\Phi^*\psi)}{\partial t} + \frac{\partial(\Phi^*\psi v)}{\partial x} = 0, \quad (26)$$

which after integration yields

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} dx \, \Phi^*\psi = 0, \quad (27)$$

whence

$$\int_{-\infty}^{+\infty} dx' \, \Phi^*(v_0,x',t)\psi(x',t)$$
$$= \int_{-\infty}^{+\infty} dx_0 \Phi^*(v_0,x_0,0)\psi(x_0,0). \quad (28)$$

Multiplying Eq. (28) by $\Phi(v_0,x,t)$, integrating with respect to v_0, and using Eq. (25), we have

$$\psi(x,t)=(m/2\pi\hbar)\int_{-\infty}^{+\infty}dv_0$$

$$\times\Phi(v_0,x,t)\int_{-\infty}^{+\infty}dx_0\Phi^*(v_0,x_0,0)\psi(x_0,0),$$

$$(29)$$

whence the propagator reads

$$K(x,x_0,t)=(m/2\pi\hbar)\int_{-\infty}^{+\infty}dv_0\Phi(v_0,x,t)\Phi^*(v_0,x_0,0).$$

$$(30)$$

With the help of Eqs. (21), (22), and (24), we have explicitly

$$K(x,x_0,t)=(m/2\pi\hbar)\int_{-\infty}^{+\infty}dv_0\left(\frac{a(t)}{a_0}\right)^{-1/2}$$

$$\times\exp\left[\left(\frac{im\dot{a}(t)}{2\hbar a(t)}-\frac{1}{4a^2(t)}\right)\right.$$

$$\times[x-X(t)]^2+\frac{im\dot{X}(t)}{\hbar}$$

$$\times[x-X(t)]\left]\exp\left[\frac{i}{\hbar}\int_0^t dt'\left(\frac{1}{2}m\dot{X}^2(t')\right.\right.$$

$$\left.\left.-V[X(t')]-\frac{\hbar^2}{4ma^2(t')}\right)\right],$$

$$(31)$$

where $X(t)$ and $a(t)$ are the solutions to Eqs. (19) and (20), subject to the initial conditions (21). From Eqs. (12) and (31), the zeroth-order quantum action (divided by $\hbar$)

$$S_0[X(t),t]=\frac{1}{\hbar}\int_0^t dt'\left(\frac{1}{2}m\dot{X}^2(t')-V[X(t')]\right.$$

$$\left.-V_{qu}[X(t')]\right)$$

$$(32)$$

shows the presence of the quantum potential [8]

$$V_{qu}[X(t)]=\frac{\hbar^2}{4ma^2(t)}.$$

$$(33)$$

Moreover, Eqs. (30) and (31) show that the relevant quantum-mechanical information for the propagator is contained in the guiding wave function. The quantum propagator can be viewed as an expansion of the guiding wave function over the v_0 space [9].

Next, we turn to the description of a (Ohmic) friction mechanism via the causal interpretation of quantum mechanics. This description is motivated by classical phenomenological models that have successfully resolved problems of, for example, a particle in a gas, without having to account for the Avogadro's number of degrees of freedom of the environment, although in a different context we have the example of the Navier-Stokes model for the description of a viscous fluid [10].

To this end, we generalize Eq. (3) to [11]

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\nu v=-\frac{1}{m}\frac{\partial}{\partial x}(V+V_{qu}),\qquad(34)$$

where ν is the friction coefficient and the third term on the left-hand side of Eq. (34) accounts for the Ohmic friction mechanism. With the help of Eq. (7), we can readily obtain

$$\frac{\partial S}{\partial t}+\nu S+\left(\frac{\hbar}{2m}\right)\left(\frac{\partial S}{\partial x}\right)^2+\frac{1}{\hbar}(V+V_{qu})=0.\qquad(35)$$

Equations (7) and (35) yield

$$\frac{d^2x}{dt^2}+\nu\frac{dx}{dt}=-\frac{1}{m}\frac{\partial}{\partial x}(V+V_{qu}).\qquad(36)$$

By following the same procedure developed to obtain Eqs. (18)–(20), we arrive at

$$\dot{S}_0+\nu S_0=\frac{1}{\hbar}\left(\frac{1}{2}m\dot{X}^2-V[X(t),t]-\frac{\hbar^2}{4ma^2}\right),\qquad(37)$$

$$\ddot{X}+\nu\dot{X}=-\frac{1}{m}V'[X(t),t],\qquad(38)$$

$$\ddot{a}+\nu\dot{a}+\left(\frac{1}{m}V''[X(t),t]\right)a=\frac{\hbar^2}{4m^2a^3},\qquad(39)$$

subject to the same initial conditions as in Eq. (21).

Likewise, the wave packet described by Eq. (1) can be written as

$$\psi(x,t)=[2\pi a^2(t)]^{-1/4}\exp\left[\left(\frac{im\dot{a}(t)}{2\hbar a(t)}-\frac{1}{4a^2(t)}\right)[x-X(t)]^2\right]$$

$$\times\exp\left[\frac{im\dot{X}(t)}{\hbar}[x-X(t)]+\frac{imv_0x_0}{\hbar}\right]$$

$$\times\exp\left[\frac{i}{\hbar}e^{-\nu t}\int_0^t dt'\right.$$

$$\left.\times e^{\nu t'}\left(\frac{1}{2}m\dot{X}^2(t')-V[X(t)]-\frac{\hbar^2}{4ma^2(t)}\right)\right].\qquad(40)$$

Because our dissipative model keeps the conservation of probability Eq. (2), Eqs. (24)–(30) hold true. Consequently, we write the dissipative propagator as

$$K(x,x_0,t)=(m/2\pi\hbar)\int_{-\infty}^{+\infty}dv_0\left(\frac{a(t)}{a_0}\right)^{-1/2}$$

$$\times\exp\left[\left(\frac{im\dot{a}(t)}{2\hbar a(t)}-\frac{1}{4a^2(t)}\right)[x-X(t)]^2\right.$$

$$\left.+\frac{im\dot{X}(t)}{\hbar}[x-X(t)]\right]\exp\left[\frac{i}{\hbar}e^{-\nu t}\int_0^t dt'\right.$$

$$\left.\times e^{\nu t'}\left(\frac{1}{2}m\dot{X}^2(t')-V[X(t')]-\frac{\hbar^2}{4ma^2(t')}\right)\right],$$

$$(41)$$

where $X(t)$ and $a(t)$ are the solutions to Eqs. (38) and (39), subject to the initial conditions (22).

The classical part of the zeroth-order quantum action (divided by $\hbar$) in the propagator above,

$$S_0^{cl}[X(t),t] = \frac{1}{\hbar} e^{-\nu t} \int_0^t dt' \; e^{\nu t'} \left(\frac{1}{2} m\dot{X}^2(t') - V[X(t')] \right), \quad (42)$$

can be contrasted to that introduced by Caldirola and Kanai [12,13],

$$S_{CK} = \frac{1}{\hbar} \int_0^t dt' \; e^{\nu t'} \left(\frac{1}{2} m\dot{X}^2(t') - V[X(t')] \right). \quad (43)$$

As pointed out elsewhere [14], this action does not describe dissipation correctly, but rather a system with a time-varying mass, since the dissipative exponential factor can be transformed away [15].

Finally, from Eqs. (2) and (34), we can construct the relation [11].

$$\frac{\partial U}{\partial t} + \frac{\partial Q}{\partial x} = -\nu \rho v^2, \quad (44)$$

where

$$U = \rho[\tfrac{1}{2}mv^2 + V + V_{qu}] \quad (45)$$

and

$$Q = vU + \frac{\hbar^2}{2m^2} \left[\sqrt{\rho} \frac{\partial^2 \sqrt{\rho}}{\partial x \partial t} - \frac{\partial \sqrt{\rho}}{\partial t} \frac{\partial \sqrt{\rho}}{\partial x} \right] \quad (46)$$

are the energy density and energy density flux, respectively.

The preceding equations demonstrate that the energy-dissipation theorem is correctly satisfied and validate the model developed in this work towards a further understanding of the role of dissipation in quantum mechanics. Temperature can enter in the formalism through a stochastic force added to the right side of Eq. (34). Via the fluctuation-dissipation theorem, a solution to Eq. (38) can be obtained and incorporated to the wave propagator (41).

Above all, this work shows that the relevant quantum-mechanical information for the propagator is contained in the guiding wave function. The quantum propagator can be viewed as an expansion of the guiding wave function over the v_0 space. Therefore, it poses an alternative route that suggests further investigations. Applications of this work are in progress and will be published in a forthcoming paper.

[1] P. R. Holland, *The Quantum Theory of Motion* (Cambridge University Press, Cambridge, England, 1993). This book presents a comprehensive account of the de Broglie–Bohm causal interpretation of quantum mechanics. See also references therein.

[2] L. de Broglie and J. P. Vigier, Phys. Rev. Lett. **28**, 1001 (1972).

[3] D. Bohm and B. J. Hiley, Phys. Rev. Lett. **55**, 2511 (1985); Phys. Rep. **172**, 93 (1989).

[4] D. Bohm, C. Dewdney, and B. J. Hiley, Nature (London) **315**, 294 (1985).

[5] A. B. Nassar, Phys. Rev. A **32**, 1862 (1985); **33**, 2134 (1986).

[6] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[7] I. B. Bernstein, Phys. Rev. A **32**, 1 (1985).

[8] The quantum potential assumes a more general form
$$V_{qu}(x) = -\frac{\hbar^2}{2m} \left[\frac{[x - X(t)]^2}{4a^4(t)} - \frac{1}{2a^2(t)} \right].$$

[9] The free-particle propagator can be obtained by using $a(t) = a_0\sqrt{1 + (t/\tau)^2}$, where $\tau = 2ma_0^2/\hbar$ and $X(t) = x_0 + v_0 t$. Useful formulas are $\exp[-i\tan^{-1}(t/\tau)] = [1 - i(t/\tau)]/\sqrt{1+(t/\tau)^2}$ and $\int_0^\infty dy \, \exp[-Ay^2 + By] = \sqrt{(\pi/A)} \exp(B^2/4A)$.

[10] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1979). See Chap. 2.

[11] A. B. Nassar, J. Math. Phys. (N.Y.) **27**, 2949 (1986).

[12] P. Caldirola, Nuovo Cimento **18**, 393 (1941); Nuovo Cimento B **77**, 241 (1983).

[13] E. Kanai, Prog. Theor. Phys. **3**, 440 (1948).

[14] J. R. Ray, Lett. Nuovo Cimento **25**, 47 (1979).

[15] A. B. Nassar, J. Math. Phys. (N.Y.) **27**, 755 (1986).